Supply Contracts Under Bounded Order Quantities

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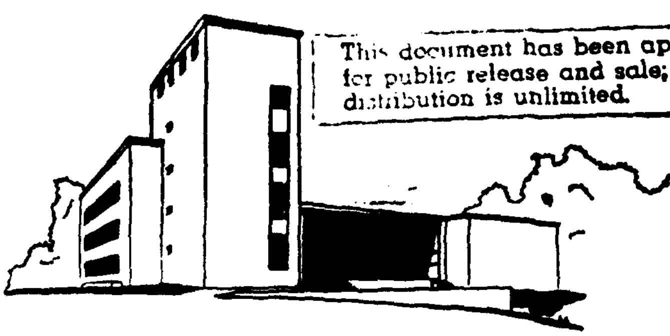

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Abstract

It is the practice in some industries, as well as within multiplant organizations, to specify bounds on the range of allowable order values. That is, the buyer contracts to place an order within a small range in a future period. The specification of the range protects the supplier against large variations in the order at short notice, although it reduces the flexibility of the buyer to respond to demand changes in his own market. Hence, in exchange, the supplier is willing to reduce the price of the component to the buyer. This paper examines the implications of this specification for the production/inventory decisions. The optimal contract is determined from the structure of the solution to the production/inventory problem.

1 Introduction

An important issue that arises in supply management is the allocation of the cost of uncertainty in demand for the final products between the buying and the supplying firms. If the buying and supplying plants are in a single firm, as in a vertically integrated organization, the production decisions of each plant have to be coordinated towards an overall objective. In this scenario, then, the question becomes on how to communicate the requirements for production, arising from the external demand, from a plant at one hierarchical level to a plant at another level, given also the uncertainties in production at each level.

In such multi-plant organizations, one scheme, that is used in practice, is to specify the requirements for components in downstream plants for future periods in terms of a range around a nominal figure. For example, in a computer manufacturing firm, the assembly plant may specify its requirement for boards as a nominal figure Q plus / minus a five percent variation. In the actual period, the buyer plant may order any number within this range. The specification of the range serves to retain some flexibility for the buyer plant in responding to fluctuations of demand around its own forecasts, while, at the same time it protects the seller plant against large variations in requirements at short notice.

A similar scheme of supply contracts is used between firms in the computer industry. The buyer, while specifying his order for a future period, contracts to buy a quantity within a range that is mutually agreed on by the buyer and the seller. It is in the interest of the supplier to minimize the width of this range, since it reduces the uncertainty in his forecasts, and hence it is the practice of some semiconductor firms to offer discounts for specification of a smaller range in the contract. The buyer is contractually obliged to later place a firm order that is within this specified range.

In this paper, a *buyer - supplier contract* is considered where the order Q for a *component* in a future period is specified in terms of a range, say U through L, which are the upper and lower bounds on the order quantity respectively. The price specified for the component in the contract is a function of the range U, L. The supplier produces the component and his problem is to determine the production quantities for the component in each period. On the other hand, the buyer has an assembly plant that assembles the component into a *product*. The problem for the buyer is to determine the order quantity for the component in each period. Given that each plant individually makes its own decisions, the overall problem is the contract, that is, the price and the order range.

The decision model of each contracting firm is a function of his information availability. It is assumed that the buyer and the supplier are aware of all unit costs incurred by each other, as well as the decision models used for production inventory decisions. Further, the buyer is fully aware, at the time of the contract, of the parameters of the demand distribution. However, the supplier is uninformed regarding the buyer's demand. (In a sequel paper (Kumar & Akella, 1991), the situation when the supplier is fully aware of the demand distribution is considered.)

In fact, in this approach to the problem, the only information available to the supplier is

the range of values the order can take in each period (which is specified in the contract). In other words, the order for each product, though time varying, is known to lie within a range that may be specified in terms of an upper and a lower bound only.

The issue of capacity constraints on the supplier's production is addressed in the model. Typically, as in the case of semiconductor or auto industries, the capacity issue is significant mainly at the component manufacturing stage, where the costs of capacity expansion are prohibitive in the short run.

In the context of the scenario described above, the following questions are addressed:

1. What is the optimal contract, that is, what are the prices and associated demand ranges that will prevail in this scenario?
2. Given a particular price and a range of order values, how do the supplier and buyer make their production decisions?

1.1 Literature Survey

The literature review is classified broadly into three categories. In the first section, research in two stage inventory problems is examined, since the analysis in this paper subsumes a two stage decentralized inventory problem. In the next section, inventory models in literature that have features similar to those in this paper are discussed. Finally, some supply management models that have recently appeared in literature are presented in the last section.

1.1.1 Two Stage Inventory Models

A fairly exhaustive survey on multi-echelon inventory theory through 1971 may be found in Clark (1972). One of the earliest models was developed by Clark & Scarf (1960) for multiple stages in series with demand at the lowest stage. Using linear cost assumptions, they developed simple optimal policies for each echelon.

However, if the problem is expanded to include multiple products, or to include parallel stages, such simple policies are no longer obtained. Eppen & Schrage (1981) study a central depot - multiple warehouse with random demands at the warehouses, for which under some restrictions, they derive approximately optimal policies. The policy for the depot is assumed to be a base stock policy, while the policies for the warehouses are more complex, in that the system inventory is raised to the base stock level once every m periods.

A similar problem is studied by Federgruen & Zipkin (1984), where they relax some of the assumptions. They approximate the allocation problem and the dynamic program to a single-location inventory problem. The policies available for such single location problems may then be applied to obtain policies here.

Schmidt & Nahmias (1985) derive optimal policies for a two-stage assembly system, in which two components ordered from external suppliers are assembled to form an end product.

1.1.2 Relevant Inventory Models

An excellent classification of research in inventory theory is provided in Silver (1981), where the criteria for classification include the form of the objective and constraints, the relevant costs, single or multiple items, nature of the demand, number of periods, stationarity of parameters, nature of the supply process, backorders or lost sales, number of stocking points and perishability of stocks. Much of the literature considers the objectives to be cost minimization thus implicitly assuming that inventory decisions do not affect revenues. The minmax (maxmin) objective (which we have used) was first introduced by Scarf (1958) in analyzing distribution-free inventory problems.

Basu (1987) considers a one-period inventory model with random supply, where the distribution of the future demand is unknown. He shows that the resulting Minmax formulation is equivalent to the assumption of an exponential distribution for the demand, and on that basis an estimate for the order quantity is developed. Azoury (1985) and Azoury & Miller (1984) study Bayesian formulations for the inventory problem, where the distribution function for the demand depends on the information already known about the demand.

Few researchers have incorporated finite production capacity in their inventory (or production) models, since the determination of optimal policies under finite capacity conditions makes the mathematical analysis considerably more difficult. Federgruen & Zipkin (1986a) consider a single item, periodic - review inventory model where the production in each period is limited to a finite capacity. The demand in each period is assumed to follow a discrete, independent and stationary distribution and can be completely backordered. They show that, rather like the infinite capacity case, the base stock policy is optimal. The optimal policy is to order up to the base stock if feasible, else to produce up to capacity. In a sequel paper (Federgruen & Zipkin, 1986b), they generalize their results to the case of continuous demand distributions under the discounted cost criterion, and here too, the modified base stock policy is shown to be optimal.

1.1.3 Contract Models

The problem of supply management, and the more specific problem of subcontracting have only recently received attention in the manufacturing literature. McMillan (1990) provides a comprehensive analysis of subcontracting as practiced by the Japanese in contrast with the American manufacturing industry. The author offers insights on several aspects of successful subcontracting relationships. Kamien & Li (1990) define subcontracting as supply sourcing of components that may be manufactured in-house. Using a two firm world, they show that the optimal contracting mechanism is a co-ordination contract. A co-ordination contract specifies prices and quantities of subcontracting that would have optimized a co-operative effort by both firms. They also illustrate the co-ordination contract in an aggregate planning model.

2 Model

The structure of the buyer-supplier relationship is illustrated in Figure 1. The buyer, whom we model as an assembly plant, supplies a product to an external market. The demand for the product is random, but is drawn from a stationary distribution. The buyer produces after demand realization, and hence places his firm orders for the component only after the demand is known. However, he specifies a range of values, represented by an upper and lower bound on the order, from which the final order will be placed in the period. The supplier, who manufactures the components, produces to stock based on the demand information given to him.

Figure 1: Structure of the Buyer - Supplier Relationship

The sequence of decisions in time is as specified below.

- Supplier and buyer negotiate the contract. The decision variables in the contract are the price of the component, P_s, and the range of values for the order (henceforth referred to as the order range) specified by U and L.
- The supplier releases his production for the period, X. The components are available for shipping at the end of the period.
- The demand for the period, D, is realized.
- The buyer observes the demand and places an order Q. He receives the shipment, produces the required number of products and ships to his customers. If the order cannot be satisfied by the supplier, it is backordered to the next period at a cost.

The notation in the analysis is listed in Appendix A.

The following assumptions are made in the model:

1. The demand for the product is drawn from a stationary distribution.
2. The buyer and the supplier possess full information regarding the decision models that each uses, and the costs incurred by each in making the decisions. However, the supplier is assumed to have no information regarding the nature of the demand distribution of the buyer.
3. The order for each component in each period is known to lie within a given range of values. Specifically, if Q_t is the demand for component in period t, then

$$\begin{aligned} Q_t &= L + \mu_t(U - L) \\ \mu_t &\in [0, 1] \end{aligned} \tag{1}$$

4. In the analysis it is assumed that the upper and lower bounds do not change over time. This is for ease of exposition only, and the results and the analysis hold for nonstationary order ranges.

5. All unit costs and prices are assumed constant over time.

In the sections below, the specific models used for the contracting decision as well as the production decisions of each firm are presented.

2.1 The Contracting Problem

The sequence of decisions leading to the contract is modeled as a leader-follower game as below:

- Buyer specifies the order range to the supplier.
- Supplier responds with the component price for that order range.

Let $\Pi_b(X^*(P_s, U, L), Q^*(P_s, U, L))$ be the expected optimal profits to the buyer if the component price is P_s, and the order range is defined by U, L. The buyer would then want to choose an order range that maximized his profits. The problem may be formulated as below:

$$\begin{aligned} \max_{U,L} \quad & \Pi_b(X^*(P_s, U, L), Q^*(P_s, U, L)) \\ st \quad & \\ & \Pi_s(X^*(P_s, U, L)) \geq \overline{\Pi} \end{aligned} \tag{2}$$

where $X^*(P_s, U, L)$ is the solution to the supplier's inventory / production problem, and $Q^*(P_s, U, L)$ is the optimal ordering decision of the buyer. Note that the profits depend on the solutions to the buyer's and supplier's production problems.

The supplier will not be willing to enter the contract unless the component price is such that he evaluates himself to be at least as well off as he was without the contract. The production decisions that the supplier and the buyer make are formulated below as subgames within the overall contracting game.

2.2 The Inventory / Production Problem

The inventory / production problems of the buyer and the supplier are also formulated as a game. The sequence of actions in each period in this game is as below:

- The supplier observes his inventory at the beginning of the period, and makes the production decision for the period.
- The buyer observes the demand and place an order quantity within the order range.

- The supplier supplies the ordered quantity, if possible. Otherwise he supplies as much as he is able to, and the rest is backordered to the next period.

Note that in this game the price of the component and the order range are exogenously specified.

2.2.1 The Buyer's Problem

The buyer observes the demand and decides on the order quantity to maximize his profits over the horizon. Note that, since he knows the decision model of the supplier he is able to compute the production quantity of the supplier. Hence, his ordering decision will be a function of the demand and the production decision of the supplier. For a demand D and the supplier production X let the profits of the buyer be denoted by $\Pi_b(X, D)$.

The buyer solves the following maximization problem:

$$\begin{aligned} \max_{Q} \quad & \Pi_b(X, D) \\ \text{s.t.} \quad & X \text{ solves the supplier problem} \end{aligned} \tag{3}$$

2.2.2 The Supplier's Problem

The supplier is required to take a production decision X given that the order quantity will lie between U and L. Recall that the order quantity may be represented by the variable μ, where

$$Q_t = L + \mu_t(U - L) \tag{4}$$

Since the supplier does not know the buyer's demand distribution, he will not be able to compute, at the time of the contract, the optimal ordering policy of the buyer. Hence, the supplier evaluates each production decision by the lowest profit obtained across all possible order quantities. He then chooses the production decision that yields the maximum lowest profit. Equivalently, if $\Pi_s(X, \mu)$ is the profit of the supplier for an order Q, and a decision X, then the supplier solves the following problem:

$$\max_{X} \min_{\mu} \quad \Pi_s(X, \mu) \tag{5}$$

The above program is referred to as the maxmin program, and any solution X as the maxmin solution.

The supplier, according to our formulation, characterizes each decision by its worst outcome. This will yield a conservative solution for him. Under the assumption of limited information about the demand, it gives the manager the assurance of a guaranteed profit level. Hence, this would be a good approach for the purposes of evaluating price reductions in the contract. Alternate formulations, which are appropriate under the assumption that the

supplier is aware of the nature of the demand for the buyer's product, have been considered in Kumar & Akella (1991).

In the sections below, solutions are developed for the problem for the single period and multiple period situations.

3 Single Period Problem

In dealing with the contracting problem, the supplier's production decision is addressed first. Based on the supplier's optimal decision, the optimal ordering policy of the buyer will be developed. This will then be used for deriving the optimal contract. Since he is unable to observe the buyer's demand distribution, the supplier will solve his problem independently with only the order range as input.

3.1 Solution to the Supplier Problem

Recalling the formulation for the supplier's general problem as below:

$$\max_{X} \min_{\mu} \quad \Pi_s(X,\mu) \tag{6}$$

Note that, as before,

$$Q(\mu) = L + \mu(U - L) \tag{7}$$

The argument to the function $Q(.)$ is dropped in what follows below. For a single period single product problem,

$$\begin{aligned} \Pi_s(X,\mu) &= P_s \min(X+E,Q) - c_s X - h(X+E-Q)^+ - s(X+E-Q)^- \\ X_t &= X \end{aligned} \tag{8}$$

where E is the initial inventory. The maxmin formulation is therefore the following:

$$\begin{aligned} \max_{X} \min_{\mu} \quad & P_s(X+E) - c_s X - (h+P_s)(X+E-Q)^+ - s(X+E-Q)^- \\ & s.t. \\ & X \le b \end{aligned} \tag{9}$$

The following two Propositions below establish the properties of the profit function. The proofs for generalized versions of the propositions are presented in the Appendix.

Proposition 1 *For a given production decision, X, the profit function $\Pi_s(X,\mu)$ is concave in μ.*

Proof: See Appendix B

Proposition 2 *The minimum value of the profit function, for a given production decision X, will be at an extreme point of the set of feasible values for μ.*

Proof: See Appendix B

Although Proposition 2 shows that the minimum profits for each decision occur necessarily at an extreme point, it does not follow that the maxmin decision should be an extreme point. From Propositions 1 and 2, it is necessary and sufficient to look at the profit at the extreme points for each production decision. The problem is reformulated as below:

$$\max_{X} \quad Z$$
$$s.t.$$
$$\begin{aligned} & X \leq b \\ & P_s(X+E) - c_s X - (h+P_s)(X+E-Q(0))^+ - s(X+E-Q(0))^- \leq Z \\ & P_s(X+E) - c_s X - (h+P_s)(X+E-Q(1))^+ - s(X+E-Q(1))^- \leq Z \end{aligned} \tag{10}$$

where

$$\begin{aligned} Q(0) &= L \\ Q(1) &= U \end{aligned}$$

The new constraints in the formulation reflect the fact that the minimum profits over μ occur at the extreme points of the feasible order values. Although the formulation is non-linear, it is easy to derive the maxmin decision.

Proposition 3 *The maxmin decision X is given by*

$$X = \begin{cases} X^* & \text{if } X_{min} \leq X^* \leq X_{max} \\ X_{min} & \text{if } X^* < X_{min} \\ X_{max} & \text{if } X^* > X_{max} \end{cases} \tag{11}$$

where

$$\begin{aligned} X_{max} &= \min(\max(U-E,0),b) \\ X_{min} &= \max(L-E,0) \\ X^* &= \min\left(\frac{sU+(h+P_s)L}{s+h+P_s} - E, 0\right) \end{aligned}$$

Proof: See Appendix B

The following properties regarding the maxmin production decision of the supplier are evident on inspection of the maxmin solution:

1. X^* is nonincreasing in the component price P_s.
2. X^* is nondecreasing in either bound of the order range, U or L.

3.2 Solution the Buyer's Ordering Problem

The buyer would like to choose an order quantity that would maximize his profits, subject to the constraint that the supplier has chosen the production quantity derived above in (11). Recalling the formulation (3) from above.

$$\begin{array}{ll} \max\limits_{Q} & \Pi_b(X, D) \\ \text{s.t.} & X \text{ solves the supplier problem} \end{array} \tag{12}$$

For the single period problem, the formulation reduces to

$$\max_{Q} \left[(P_b - c_b)\min(D, Q, X) - h_b(\min(Q, X) - D)^+ - s_b(D - \min(Q, X))^+\right] \tag{13}$$

where X is the maxmin production of the supplier and $L \leq Q \leq U$.

In the single period problem, this order policy is trivial. It is evident on inspection that the optimal order policy of the buyer is to order such that $Q = \max(\min(D, U), L)$.

3.3 Solution to the Contracting Decision

Let the initial conditions in the scenario be given by $U = U_0$; $L = L_0$; $P_s = P_0$. These conditions exist before the contracting or negotiating process and are presumably set when the supplier is selected for supply of the component.

Finally, in order to derive a specific contract, it is assumed that the buyer changes the order range symmetrically about the mean. Therefore, we have

$$\gamma = (L_0 + U_0)/2$$

$$a_0 = (U_0 - L_0)/2$$

$$L = \gamma - a$$

$$U = \gamma + a$$

where a is the variable under the buyer's control.

It is assumed, for convenience, that the initial inventory is zero. The presence of an initial inventory does not affect the analysis or solution materially.

3.3.1 Feasible Set of Prices for Supplier

If the buyer specifies an order range a, the supplier will respond with a price P_s that will ensure that he gets a minimum profit at least as much as the minimum profit under the

initial conditions. Equivalently, this results in a constraint to the buyer's profit maximization problem

$$\Pi_s(P_s, a) \geq \Pi_s(P_0, a_0) \tag{14}$$

In the following analysis, only the case when the solution to the supplier problem lies between X_{min} and X_{max} is considered. If the solution is at either extreme, the contracting decisions may be obtained in a straightforward manner. We recall the supplier's maxmin production decision

$$\begin{aligned} X &= \frac{s(\gamma + a) + (h + P_s)(\gamma - a)}{s + h + P_s} \\ &= \gamma + a\frac{s - h - P_s}{s + h + P_s} \end{aligned} \tag{15}$$

The corresponding maxmin profits are

$$\Pi_s(P_s, a) = (P_s - c_s)\gamma - P_s a - ha\left(\frac{2s}{s + h + P_s}\right) - c_s a\left(\frac{s - h - P_s}{s + h + P_s}\right) \tag{16}$$

Incorporating the above in inequality (14), the following quadratic inequality is obtained:

$$P_s^2(\gamma - a) + P_s[(s + h - c_s)(\gamma - a) - c_0] - [2sha + c_s a(s - h) + (s + h)(c_s\gamma + c_0)] \geq 0 \tag{17}$$

where

$$c_0 = P_0(\gamma - a_0) - ha_0\left(\frac{2s}{s + h + P_0}\right) - c_s\left(\gamma + a_0\frac{s - h - P_0}{s + h + P_0}\right) \tag{18}$$

Solving (17) will give a lower bound (on the price) as a function of the order range, a. The buyer will incorporate the constraint that the P_s will be at least as much as the lower bound. Note that

$$2sha + c_s a(s - h) + (s + h)(c_s\gamma + c_0) = 2sha + c_s as + h(c_s\gamma + c_0) + c_s h(\gamma - a) \tag{19}$$

and hence is strictly positive for positive cost parameters. Since the coefficient to P_s^2 is always nonnegative, the solution to (17) will always have one nonpositive root which may be ignored.

Hence, we have

$$\mathcal{P} = \{P_s : P_s \geq P_{min} \text{ where } P_{min} \text{ solves (17)}\} \tag{20}$$

3.3.2 Supply Contract

The contract will therefore be a solution to the buyer's maximization problem:

$$\begin{aligned} \max_{P_s, a} \quad & \Pi_b(P_s, a) \\ \text{s. t.} \quad & \\ & P_s \geq P_{min} \end{aligned} \tag{21}$$

The buyer's expected profits, for a given order range will be shown to be decreasing in the component price. Hence he will set the component price at P_{min}. Then, it is shown that the buyer's profits are increasing with decreasing order range under some conditions. These conditions will therefore dictate whether a contract will be negotiated.

In Section 3.2, for a demand D, the buyer's optimal policy was shown to be to order $Q = \max(\min(D,U),L)$. At the beginning of the period, therefore, the buyer's expected profits would be

$$\begin{aligned}\Pi_b(P_s,U,L) \;=\; & (P_b-c_b)\left[\int_0^X Df(D)dD+\int_X^\infty Xf(D)dD\right]\\ & -P_s\left[\int_0^L Lf(D)dD+\int_L^X Df(D)dD+\int_X^\infty Xf(D)dD\right]\\ & -h_b\int_0^L(L-D)f(D)dD\\ & -s_b\int_X^\infty(D-X)f(D)dD \end{aligned} \tag{22}$$

The terms in the above expression refer to the expected revenues from product sales, expected purchases of components from the supplier, expected holding costs for excess components ordered due to the minimum order quantity, and expected shortage costs, either due to shortage in supply or due to the upper bound on the order.

Proposition 4 *The buyer's expected profits, $\Pi_b(P_s,a)$, are decreasing in P_s, for a given order range, a.*

Proof: See Appendix B

The buyer will therefore set the price at P_{min}. The buyer's problem is then to optimize his profits over the order range.

$$\max_a \quad \Pi_b(a) \tag{23}$$

A stationary point analysis will yield the optimal value of a.

It is of interest to look at the conditions under which the expected profits of the buyer are decreasing in a. In this case, the buyer will choose to order a prespecified amount from the supplier, thus incurring all the costs of variability in demand. Examining the marginal change in the expected profits with respect to the order range, we get

$$\frac{\partial\Pi_b}{\partial a}=\frac{\partial X}{\partial a}\left[(P_b-c_b-P_s+s_b)(1-F(X))\right]-\frac{\partial P_s}{\partial a}\left[E(Q)\right]-(P_s+h_b)F(L) \tag{24}$$

where, $E(Q)$, the expected order quantity is given by

$$E(Q)=\left[\int_0^L Lf(D)dD+\int_L^X Df(D)dD+\int_X^\infty Xf(D)dD\right] \tag{25}$$

and

$$\frac{\partial X}{\partial a} = \frac{s-h-P_s}{s+h+P_s} - \frac{\partial P_s}{\partial a}\frac{2as}{(s+h+P_s)^2} \tag{26}$$

From the above two equations, if for all a

$$\frac{\partial P_s}{\partial a}\left[E(Q) + \frac{2as(P_b - c_b - P_s + s_b)(1-F(X))}{(s+h+P_s)^2}\right]$$
$$\geq (P_s + h_b)F(\gamma - a) + \frac{(s-h-P_s)(P_b - c_b - P_s + s_b)(1-F(X))}{s+h+P_s} \tag{27}$$

$$\text{then } \frac{\partial \Pi_b}{\partial a} \leq 0 \tag{28}$$

The left side of Inequality (27) represents the marginal losses to the buyer for a increase in a, while the right side represents the marginal gains. The losses accrue from

1. The price increase that the supplier will charge on the expected order quantity.
2. A marginal decrease in supplier production. The decrease in supplier production will be significant only if the buyer's demand is high enough, and hence this marginal loss is weighted by $(1 - F(X))$.

The gains are due to

1. A decrease in purchase and holding costs due to a larger order range. The marginal gain is weighted by the probability that the demand is smaller than the smallest possible order.
2. An increase in the supplier production, resulting in net gains to the buyer, if his demand is higher that the supply.

If the net marginal gains from an increase in a are smaller than the net marginal losses, the buyer will operate at $a = 0$.

3.4 An Example

In this section, the contract for a given buyer-supplier situation is derived. Consider the following data:

Supplier			Buyer		
P_0	=	5	P_b	=	25
c_s	=	3	c_b	=	5
h	=	1	h_b	=	6
s	=	30	s_b	=	15

The buyer is assumed to have a product demand that is uniformly distributed between 70 and 130 units ($\gamma = 100, a_0 = 30$).

Supplier Decisions

The maxmin solution for a contract (P_s, a) is

$$X = 100 + a\frac{29 - P_s}{31 + P_s} \tag{29}$$

The solution $\mathcal{P}_{min}$ is determined as below and illustrated in Figure 2:

$$\mathcal{P}_{min} = \left(-28 + \left(-4\left(147 - \frac{22140}{100 - a}\right) + \left(28 - \frac{60}{100 - a}\right)^2\right)^{0.5} + \frac{60}{100 - a}\right)/2 \tag{30}$$

As expected, the component price decreases monotonically with the order range. From the

Figure 2: The Minimum Component Price for each Order Range

figure, it is seen that the price is convex in the order range, indicating that the supplier gains more from a reduction in the order range at high values of order range.

Buyer Decisions

The optimal expected profits of the buyer as a function of the order range a, given that the supplier will charge $\mathcal{P}_{min}$, is illustrated in Figure 3. The buyer's optimal profits can be

Figure 3: The Optimal Profits of the Buyer

observed to be concave in a, hence, the supply contract will be at the value of component price and order range that maximizes the buyer profits, which is $(P_s = 4, a = 18)$. The concavity emerges due to the fact that the supplier production decreases with increasing order range. Initially, the gains from the price reduction dominate and the buyer profits increase with reduction in the order range. However as the order range is further reduced, the losses due to a reduction in the supplier production dominate over the gains and the buyer profits fall. The contract will be at the point where the two effects exactly offset each other.

3.5 Comparison between the Maxmin and Expected Profit Solutions

An alternate model for the uninformed supplier is to assume that the order quantity is drawn from a uniform distribution. The supplier believes that every value for the order between the upper and lower bounds on the order range is equally likely. Under this assumption, and using the notation defined earlier, the production decision for the period is given by

$$X_{uniform} = \frac{U(P_s - c_s + s) + L(h + c_s)}{P_s + h + s} \tag{31}$$

Recall that the maxmin solution [1]

$$X_{maxmin} = \frac{Us + L(P_s + h)}{P_s + h + s} \tag{32}$$

- *Remark 1:* $X_{uniform} \geq X_{minmax}$ for all parameter values. The difference in the two solutions is given by

$$\frac{(P_s - c_s)(U - L)}{s + P_s + h} \tag{33}$$

  If the order range and the contribution margin are low, the two solutions are close to each other. Typically, $(P_s - c_s)/P_s$, the contribution margin is about 20%, and we expect the maximum order range, $(U - L)$, to be about 10% of the supplier production. Hence, the difference in the two solutions would be of the order of 2% of the supplier production.

- *Remark 2:* $X_{uniform}$ increases with an increase in the component price, while X_{maxmin} decreases with an increase in the component price. Further, an increase in the production cost c_s does not affect X_{maxmin} at all.

- *Remark 3:* It is easily inferred that X_{maxmin} will be a better solution if most of the orders are at one extreme or the other, and hence when the order range is small.

4 Multi-Period Problem - Uncapacitated Supplier

As in dealing with the single period problem, the problem of the supplier is addressed first. Here too, the supplier, since he is not informed about the demand distribution for the buyer's product, will solve his problem independently. The optimal ordering policy of the buyer, based on the supplier's production policy, is then addressed. Finally the contracting problem is discussed. Unlike the single period case, a closed-form solution for the component price cannot be obtained. Therefore, a constructive solution for the contract is presented.

In a multiple period problem, the supplier is assumed to make each production decision based on the evaluation of the minimum profits over the current and remaining periods. This may be formulated in recursive fashion as a dynamic program, to obtain the optimal policy for the supplier. It is shown in the section below, that, similar to expected profit formulations (when the distribution is known), the optimal policy is of the simple base-stock form. Equivalently, it is optimal for the supplier, in each period, to produce such that his initial stock is raised to the base-stock number.

The base-stock numbers may be regarded as soft capacity constraints for the buyer. That is, in each period, the buyer wil at most order receive the base-stock number. As asserted before, the buyer places his order only after he observes the demand in each period. However, unlike the single period problem, this problem is complex since the buyer may deviate from a

[1] Note that the above comparisons are valid only when the solutions are at the bounds. If not, the comparison is straightforward.

myopic policy (which is to order only his current demand) to account for future shortages or stocks resulting from the base-stock as well as the bounds on the orders. Again, as shown in the section below, a critical number policy is optimal for the buyer. The buyer should place an order that exceeds the demand for that period by the critical number value.

The algorithmic nature of the solutions for the buyer as well as the supplier precludes the possibility of obtaining analytical closed form solutions for the contract. However, the solution procedure, given problem parameters, is sketched out in the sections below.

4.1 Solution to the Supplier Problem

At the beginning of each period the supplier will choose a production decision based on the initial inventory to maximize the minimum profits obtained till the end of planning horizon of n periods. Assigning numbers 1 through n for the periods in the horizon, the recursive expressions for the supplier's problem are as below.

$$\begin{aligned} f_t(I_{t-1}) &= \max_{X_t} \min_{\mu_t} [K(I_{t-1}, X_t, \mu_t) + f_{t+1}(I_t)] \\ f_n(I_{n-1}) &= \max_{X_n} \min_{\mu_n} [K(I_{n-1}, X_n, \mu_n)] \end{aligned} \tag{34}$$

where

$$\begin{aligned} I_t &= I_{t-1} + X_t - Q(\mu_t) \\ K(I_{t-1}, X_t, \mu_t) &= P_s \min(X_t + I_{t-1}^+, Q(\mu_t) + I_{t-1}^-) \\ &\quad - c_s X_t - h I_t^+ - s I_t^- \\ \text{subject to} & \\ & X_t \geq 0, \forall t \end{aligned} \tag{35}$$

In each period the supplier's objective is to maximize the minimum profits. The profits include the period profits as well as the maxmin profits in the subsequent periods. Note that the excess demand is completely backordered in each period. Further, in the last period, only that period's profits are considered.

At the beginning of the horizon, the supplier will solve the following for the first period production decisions.

$$\begin{aligned} f_1(I_0) &= \max_{X_1} \min_{\mu_1} [K(I_0, X_1, \mu_1 + f_2(I_1))] \\ \text{subject to} & \\ & X_t \geq 0, \forall t \end{aligned} \tag{36}$$

The maxmin profits for the horizon are therefore given by the above, which represents the maxmin problem for the supplier in the first period.

Let Y_t represent the stock level of the supplier after the production has been realized. Hence $Y_t = I_{t-1} + X_t$. The recursive equations become:

$$\begin{aligned} f_t(I_{t-1}) &= \max_{Y_t} \min_{\mu_t} \left[L(Y_t, \mu_t) + f_{t+1}(Y_t - Q(\mu_t)) \right] \\ f_n(I_{n-1}) &= \max_{Y_n} \min_{\mu_n} \left[L(Y_n, \mu_n) \right] \end{aligned} \tag{37}$$

In the proposition below, the base stock policy is shown to be optimal in this case.

Proposition 5 *There exist numbers $S_t, t = 1, \ldots, n$, such that $f_t(I_{t-1})$ is maximized for $Y_t = S_t$.*

Proof: See Appendix B

In each period, therefore, the optimal policy for the supplier is to raise his initial stock to the critical number. In this regard, the policy is similar to that obtained when maximizing expected profits under stochastic demand conditions, which is a well known result in stochastic inventory theory. The critical number policy results in the maxmin case because of the concavity property of the value functions. However, this property is difficult to establish inductively due to the fact that the one period profit and the profit-to-go functions are not separable under the *min* operator.

It may be noted from the proposition, that, at the optimal policy in each period, the profit at the lowest realization of demand is equal to the profit at the highest realization of demand. This property was also observed in the single period case and it facilitates the computation of the critical number in each period.

In the next section, the computation for a lower bound on the supplier profits is presented. The computation of the critical numbers may prove cumbersome for some cases, and hence the lower bound may be used as an approximation for the supplier profits. Although the computation of the lower bound is in itself a hard problem, it is easier to solve for than the critical numbers.

As shown below, the buyer's problem is simplified considerably due to the critical number policy of the supplier, so that policies independent of the supplier's actual production or stock levels may be obtained.

4.2 Solution to the Buyer's Problem

The buyer chooses an ordering policy that maximizes his expected profits over the horizon. Recall that the buyer is aware of the decision making models of the supplier, and hence is able to account for the decisions in his own decision making model. The buyer solves the following problem (3)

$$\begin{aligned} \max_{Q} \quad & \Pi_b(X, D) \\ \text{s.t.} \quad & X \text{ solves the supplier problem} \end{aligned} \tag{38}$$

If the supplier could, for each period in the horizon, fulfill any order that the buyer may place, then the ordering policy of the buyer is trivial. In each period, the buyer orders a quantity exactly equal to the demand, or equivalently, $Q_t = D_t$. In general, this is not realistic, however, it defines the myopic policy Q_t^m.

$$Q_t^m = \min(\max(D_t, L), U) \tag{39}$$

If the supplier uses a base-stock policy, then the optimal production policy of the buyer is a critical number policy that is different from the myopic policy.

The buyer is assumed to take the following sequence of actions in each period t:

- After he sees the demand for the period, he orders a quantity Q_t from the supplier, where $U \geq Q_t \geq L$.
- The supplier can at most supply S_t. He backorders the quantity $[Q_t - S_t]^+$.
- The buyer backorders $[D_t - S_t]^+$ where D_t may be larger than Q_t. At the end of the period, the buyer places a firm order of $[D_t - Q_t]^+$ with the supplier. Since the supplier has a large capacity, the supplier is able to supply this additional order in each period. Note that this sequence of firm orders does not affect the supplier's stocking decisions and may be treated separately.
- The buyer, therefore, can meet his backorders completely in the next period.

We introduce the following additional notation

B_t : Inventory level at the beginning of period t.

Z_t : Inventory level of the buyer after arrival of the order from the supplier.

Hence,

$$B_t = Z_{t-1} - D_{t-1} \tag{40}$$

The supplier has unlimited production capacity, and from Proposition 5 we know that the supplier will implement a base stock policy. The base stock of the supplier is therefore a constraint on the buyer, since the buyer will not produce more than what can be supplied in that period.

The recursive equations for the buyer's problem are

$$\begin{aligned} f_t(B_t) &= \max_{Z_t \geq B_t} J(Z_t, D_t) + E\left[f_{t+1}(Z_t - D_t)\right] \\ f_{n+1}(B_{n+1}) &= 0 \end{aligned} \tag{41}$$

where

$$J(Z_t, D_t) = (P_b - c_b)\min(D_t, Z_t) - P_s(Z_t - B_t) - h_b(Z_t - D_t)^+ - s_b(Z_t - D_t)^- \tag{42}$$

An accounting change is made here for convenience. Note that it is known that the backorder B_{t+1}^- will be satisfied in the next period. The charges associated with the backorder, namely the revenue $(P_b - c_b)B_{t+1}^-$ and the purchase cost $P_s B_{t+1}^-$, are brought forward to period t. Hence

$$J(Z_t, D_t) = \left[(P_t - c_b)D_t - P_s(Z_t - B_t^+) - h_b(Z_t - D_t)^+ - (s_b + P_s)(Z_t - D_t)^-\right] \quad (43)$$

The analysis presented below is not in any way affected by this change.

In each period, the buyer chooses an ordering decision to maximize the profits for the period and the expected profits for the remaining periods.

The following constraints apply in each period

$$\begin{aligned} L \le Z_t - B_t^+ &\le U \\ Z_t - B_t^+ &\le S_t \end{aligned} \quad (44)$$

The first constraint ensures that the order in each period is within the bounds, while the second constraint states that the order must be less than the base-stock of the supplier.

The optimal ordering policy for this scenario is now shown to be similar to a base-stock policy. However, here, since the demand is known the order will be a function of the demand. More particularly, the optimal policy has a simple linear form.

Proposition 6 *There exist numbers $R_t, t = 1, \ldots, n$, such that*

$$Z_t = R_t + D_t \quad (45)$$

maximizes $f_t(B_t), t = 1, \ldots, n$.

Proof: See Appendix B

The optimal policy therefore will be given by

$$Z_t^* = \min(S_t, \max(L, D_t - B_t^+ + R_t), U) \quad (46)$$

Further the optimal ordering policy will be

$$Q_t^* = \min(U, \max(D_t - B_t^+ + R_t, L)) \quad (47)$$

(Note that $R_t = 0$ yields a myopic ordering policy). Using this policy, the buyer's expected profits for the horizon may be computed.

According to the optimal policy the buyer ensures production of the demand plus a safety stock in case he falls short in future periods, due to either the bound on the order or the stocking policy of the supplier. However, he orders as much as he needs (within the bounds) so that the supplier will backorder his demand to the next period.

4.3 Solution to the Contracting Decision

The solutions to the ordering decision of the buyer and the inventory decision of the supplier are algorithmic in nature. This precludes the possibility of obtaining an analytical solution to the contracting problem, and hence a constructive solution through the use of graphs is presented.

The solution procedure remains the same as for the single period problem. The first step is to obtain the price of the component as a function of the order range. It is clear that the buyer will force the supplier to operate at the minimum possible price. In the next step, the buyer's profits are found as a function of the order range, given that the supplier's component price is the minimum price. The buyer then chooses the order range that maximizes his own profits.

The constructive procedure is illustrated using a two-period example. Consider the following data:

Supplier			Buyer		
P_0	=	5	P_b	=	18
c_s	=	3	c_b	=	10
h	=	1	h_b	=	1
s	=	30	s_b	=	30

The buyer is assumed to have a product demand that is uniformly distributed between 70 and 130 units ($\gamma = 100, a_0 = 30$) for each of the two periods considered. We further assume that the supplier and the buyer have no initial inventories.

Supplier Decisions In this problem, where there is no initial inventory and a low cost of component manufacture, the order upto decisions in both periods are nearly identical and equal to

$$S_1 = S_2 = \gamma + a\frac{s - P_s - h}{s + h + P_s} \tag{48}$$

The corresponding Maxmin profits for the supplier over the two periods are

$$\Pi_s = 2P_s(\gamma - a) - 2c_s(\gamma - a\frac{P_s + h}{s + h + P_s}) - 4ah\frac{s}{P_s + h + s} \tag{49}$$

Although, in this particular problem, a closed form expression for the minimum price may be easily obtained (by setting $\Pi_s = \Pi_0$, where Π_0 are the maxmin profits under initial conditions), in general, it may not be possible to do so. Hence, we will illustrate the constructive solution rather than avail of the expression for the minimum price.

The profits for a large number of order ranges and component prices are first evaluated. Subsequently, given the initial maxmin profits we may derive the minimum price curve, by reading off the associated order ranges for various component prices. This curve is constructed in Figure 4. As in the single period case, the minimum price curve is convex. Hence, a reduction in uncertainty is more valuable to the supplier when the uncertainty is high.

Buyer Decisions It is possible now to evaluate the buyer policies and hence his expected profits for a number of price - order range combinations. Given a order range *a* and the price

Figure 4: Minimum Supplier Price

range P_s, the optimal critical numbers are as below:

$$\begin{aligned} R_2 &= 0 \\ s - P_s - h &= hF(L+R_1) - P_sF(L+R_1) + P_sF(R_1+S) + sF(S+R_1) \end{aligned} \tag{50}$$

This yields the following expected profits

a	P_s	S_1	S_2	R_1	R_2	Profits
0	3.50	100.0	100.0	19.35	0	467
3	3.65	102.2	102.2	16.42	0	456
6	3.81	104.3	104.3	13.48	0	426
9	3.97	106.4	106.4	10.53	0	378
12	4.15	108.5	108.5	7.51	0	306
15	4.34	110.5	110.5	4.52	0	212
18	4.54	112.4	112.4	1.48	0	90

The above computations are illustrated as a graph of the buyer profits as a function of the order range in Figure 5.

Figure 5: The Optimal Profits of the Buyer

The profits peak at an optimal order range of 0 units. The supply contract therefore will be set at ($P_s = 3.5, a = 0$).

5 Multi-Period Problem - Capacity Constraints

In this section, the production problems of the buyer and the supplier are examined, when there are capacity constraints on the supplier production. The contract, as before, may be solved by numerical computation.

The introduction of capacity constraints poses two difficulties in the analysis. The first is in the supplier problem. Although a simple policy may be shown to be optimal for the supplier, the computation of the critical numbers is hard due to the capacity constraints. An alternative approach is to compute a lower bound on the supplier profits for the purposes of the contract. The second difficulty lies in the analysis of the buyer's problem. Since the stock levels of the supplier in each period are no longer independent of the initial inventory in each period, the 'capacity constraint' on how much the supplier can deliver is now a random variable and cannot be computed a priori. Hence, a simple critical number policy cannot be easily obtained in the buyer's problem.

As in the last section, the supplier's problem is analyzed first. A lower bound on the supplier's profits is suggested in the next subsection, and finally the buyer's problem is addressed.

5.1 Solution to the Supplier Problem

The notation developed in Section 4 is used here and is also detailed in Appendix A. The following recursions represent the supplier's problem.

$$\begin{aligned} f_t(I_{t-1}) &= \max_{Y_t} \min_{\mu_t} \left[K(Y_t, \mu_t) + f_{t+1}(Y_t - Q(\mu_t))\right] \\ f_n(I_{n-1}) &= \max_{Y_n} \min_{\mu_n} \left[K(Y_n, \mu_n)\right] \end{aligned} \tag{51}$$

where

$$K(I_{t-1}, X_t, \mu_t) = P_s \min(Y_t, Q(\mu_t) + I_{t-1}^-) - c_s(Y_t - I_{t-1}) - hI_t^+ - sI_t^- \tag{52}$$

Further, there is a capacity constraint in each period t

$$X_t \leq b \tag{53}$$

And, as before

$$Y_t = X_t + I_{t-1} \tag{54}$$

In the proposition below, the policy is derived for the supplier's problem. It is shown that the modified base stock policy is optimal. In this policy, the supplier raises his initial stock in the period to the critical number, if possible, else he produces at his capacity.

Proposition 7 *There exist numbers $S_t, t = 1, \ldots, n$, such that $f_t(I_{t-1})$ is maximized by $Y_t = \min(S_t, I_{t-1} + b)$.*

Proof: See Appendix B

In each period, therefore, the optimal policy of the supplier is to produce such that the stock level is raised to the base stock. If the capacity is insufficient then the policy of the supplier is to produce up to capacity. This policy is termed as the modified base stock policy and was shown to be an optimal policy for the stochastic finite horizon inventory model (Federgruen & Zipkin (1986b)). As in the uncapacitated case, therefore, the maxmin policy under capacity constraints is similar to the policy under capacity in stochastic inventory. Once again, the proof results due to the concavity of the value function in each period, and as in the uncapacitated case, is difficult to establish due to the fact that the profit-to-go and the one period profit functions are not separable.

The computation of the critical numbers is cumbersome in this case due to the capacity constraint. As an alternative to computing the critical numbers, an approximation for the supplier's maxmin profits may be obtained from the computation of a lower bound on the maxmin profits. In the section below, the procedure for the computation of a lower bound is outlined.

5.2 A Lower Bound on the Solution to the Supplier Problem

The lower bound on the supplier profits is considered in this section. Consider the following formulation

$$\max_{X_1,\cdots,X_n} \min_{\mu_1,\cdots,\mu_n} \quad [K(I_0, X_1, \mu_1) + \cdots + K(I_{n-1}, X_n, \mu_n)] \tag{55}$$

subject to

$$X_t \leq b, \forall t$$
$$X_t \geq 0, \forall t$$
$$I_t = I_{t-1} + X_t - Q(\mu_t) \tag{56}$$

Proposition 8 *Let $F(X^*)$ be the optimal objective function value to Formulation (55 - 56). Let $G(X')$ be the optimal objective function value to the Formulation (51). Then $F(X^*) \leq G(X')$.*

Proof: See Appendix B

Therefore, $F(X^*)$ is a lower bound on the optimal maxmin profits of the supplier. Propositions 9 and 10 establish properties that will be useful to compute this lower bound.

Proposition 9 *The function*

$$F(X, \mu) = \sum_{t=1}^{n} K(I_{t-1}, X_t, \mu_t) \tag{57}$$

is concave in μ for any given X.

Proof: See Appendix B

A direct consequence of the concavity property is that the minimum value of the profit function across all realizations of the demand will occur at an extreme point of the hypercube $[0, 1]^n$.

Proposition 10 *The minimum value of the function $F(X, \mu)$ for any given X occurs when μ lies at an extreme point.*

Proof: See Appendix B

Define Φ to be the set of extreme points. Proposition 10 allows us to reformulate the problem as a maximization problem with additional constraints using the extreme points.

$$\max_{X} \quad z$$

subject to

$$F(X, \mu^j) \geq z \text{ for } j = 1, \cdots, |\Phi|$$
$$X_t \leq b, \forall t$$
$$X_t \geq 0, \forall t \tag{58}$$

The formulation above is nonlinear because the profit function, $F(.)$, is nonlinear, as shown below.

$$
\begin{aligned}
F(X,\mu^j) &= \sum_t P_s \min\left(X_t + I_{t-1}^+, Q\left(\mu_t^j\right) + I_{t-1}^-\right) - c_s X_t - hI_t^+ - sI_t^- \\
&= \sum_t P_s \left(Q\left(\mu_t^j\right) + I_{t-1}^-\right) - c_s X_t - hI_t^+ - (s+P_s)I_t^- \\
&= \sum_t \left[P_s Q\left(\mu_t^j\right) - c_s X_t - hI_t^+ - sI_t^-\right] - P_s I_n^- \qquad (59)
\end{aligned}
$$

The formulation is linearized by recursively defining the following new variables E_t, S_t, to replace the nonlinear variables I_t^+, I_t^-.

$$
E_t - S_t = X_t + E_{t-1} - Q(\mu_t) - S_{t-1}, \forall t \qquad (60)
$$

Using the above variables we get the following formulation:

$$
\begin{aligned}
\max_X \quad & z \\
& \text{subject to} \\
& \sum_t \left[P_s Q\left(\mu_t^j\right) - c_s X_t - hE_t^j - sS_t^j\right] - P_s S_n^j \geq z, \forall j \\
& E_t^j - S_t^j = X_{it} - Q\left(\mu_t^j\right) + E_{t-1}^j - S_{t-1}^j, \forall t, j \\
& X_t \leq b, \forall t \\
& X_t, S_t^j, E_t^j \geq 0, \forall t, j \qquad (61)
\end{aligned}
$$

The formulations (55 - 56) and (61) are equivalent. To show this the Proposition below is proved first.

Proposition 11 *In the optimal solution to the linear program defined by (61), the following statements hold for at least one $j, j = 1, \cdots, |\Phi|$.*

$$\textit{Either } E_t^j = 0 \quad \textit{or} \quad S_t^j = 0, \forall t$$

Proof: See Appendix B

Proposition 12 *Formulations (55 - 56) and (61) are equivalent.*

Proof: See Appendix B

The linear programming formulation (61) requires complete enumeration of the constraints. Therefore, an implicit enumeration algorithm is described below:

Solution Procedure Define $\mathcal{J}$ as the set of extreme points associated with the constraints to the linear program. The constraints associated with each extreme point μ^j, and henceforth labeled as extreme point constraints, are those described as the first two constraints in the formulation (61).

1. Initialize $\mathcal{J} = \phi$ (null set).

2. Select $\mu = [1, 1, \cdots, 1]$. Assign μ to $\mathcal{J}$.

3. Solve formulation 61 with the extreme point constraints corresponding to all $\mu^j \in \mathcal{J}$. Let X^* be the solution to the linear program.

4. Solve the following formulation.
$$\min_{\mu} F(X, \mu) \tag{62}$$
Let μ^* be the solution. By Proposition 10, μ^* must be an extreme point. If $\mu^* \in \mathcal{J}$, then stop, since X^* must be the maxmin solution. Else assign μ^* to $\mathcal{J}$ and go to Step 3.

The solution procedure will terminate in at most $|\Phi|$ iterations. In actual practice, the iterations taken are very few. For large backlog costs in comparison to holding costs, the extreme point $[1, 1, \cdots, 1]$ is a very good initial point as it is very likely that the minimum profits for the maxmin solution will lie at this extreme point.

The formulation (62) is a non-linear program, and involves the unconstrained minimization of a concave function. The problem may be solved in one of two ways.

1. General purpose algorithms developed in the literature for the minimization of concave functions (Benson (1985), Rosen (1983), etc.).

2. Using the fact that the objective function is piece-wise linear, this non-linear program may be converted into an integer program by introducing binary variables and additional constraints. The resulting integer program may be solved by a standard linear integer programming package.

The solution procedure has been tested for a few sample problems, and the results are presented in Table 1. In each problem size, 36 different problems with varying cost parameters were used. The constraints (or extreme points) generated in each run were averaged for the 36 problems and are reported in the table. The run time for each of the above problems was in the order of a few minutes. Although the lower bound has been computed for a single product case, the analysis extends easily to a multi-product case. In fact, the nonlinear function (62) is separable in products. Hence instead of solving one large integer program, m smaller integer programs are solved. In most practical cases, the number of periods n is likely to be small as compared to the number of products m, hence, the resulting computation may not be severe. A few problems with multiple products were also tested and these are reported in the table above.

5.3 Solution to the Buyer's Problem

In Section 4, it was shown that the buyer's problem is simplified considerably due to the supplier's base stock policy. This simplification does not result when the supplier has a

Problem Size (Products $\times$ Periods)	Avg No of Constraints
1×1	2.000
1×2	2.080
1×3	2.333
10×1	2.000
20×1	2.000
5×2	6.250
10×2	6.861

Table 1: Performance of the Solution Procedure

capacity constraint. Although the buyer is aware of the supplier's capacity as well as the critical numbers in the supplier's problem, the stock level of the supplier prior to order receipt is not known. In fact this stock level may be a random variable depending on the intial stock of the supplier. The buyer's assembly is therefore restricted by the supplier's random stock level.

Using the notation developed earlier in Section 4, the recursions may be used to represent the buyer's problem.

$$\begin{aligned} f_t(B_t) &= \max_{Z_t \geq B_t} J(Z_t, D_t) + E\left[f_{t+1}(Z_t - D_t)\right] \\ f_{n+1}(B_{n+1}) &= 0 \end{aligned} \tag{63}$$

where the one period costs are given by

$$J(Z_t, D_t) = (P_b - c_b)\min(D_t + B_t^-, Z_t) - P_s(Z_t - B_t) - h_b(Z_t - D_t)^+ - s_b(Z_t - D_t)^- \tag{64}$$

Further, the following constraints hold in each period

$$\begin{aligned} Z_t - B_t &\leq b + I_{t-1}^+ \\ Z_t - B_t &\leq S_t + I_{t-1}^- \end{aligned} \tag{65}$$

The first constraint above ensures that the supplier cannot deliver more than his capacity plus his initial stock. The second constraint limits the supply to the critical number (plus the backlog of the supplier). Note that the right side of the constraints are random variables since the stock of the supplier is a function of the buyer's demands. Hence it is difficult to show that a simple critical number policy is optimal here.

It is assumed, under these conditions, that a critical number policy is followed by the buyer. Then, as in the uncapacitated case, the optimal critical numbers may be numerically computed by the dynamic program. The stocking policy as well as the ordering policy follow in the same manner from the critical numbers.

6 Conclusions

In this paper, the problem of reducing the risk to the supplier through appropriate discounts in the component price was examined. In particular, the optimal contract for the single period problem, as well as the appropriate inventory/production decisions, were derived. In the multiple period problem, optimal solutions to the production / inventory problems were obtained. Using these solutions, a good contract may be numerically computed, as shown in the respective sections.

An extension to this problem is also considered, which is the inclusion of the capacity of the plants in the analysis. Typically, component manufacturers suffer limitations in capacity, since addition of capacity is normally very expensive. It was shown that a modified base-stock policy is optimal for the supplier under capacity restrictions. However, such a policy makes the analysis of the buyer's problem fairly difficult. Firstly, the buyer is restricted by the supplier's stocks, which now depend on the initial inventories to some extent. Secondly, we can no longer make the assumption that the supplier can meet all of the buyer's backorder in the next period. A possible route to solving this problem is to assume a critical number policy for the buyer, and solve for the optimal critical values.

In this paper, it was assumed that the supplier is unaware of the buyer's demand distribution. While this is true of some industries like the electronics and the computer industries, the assumption may not hold in other industries. In a sequel paper, we have looked at the problem when both the supplier and the buyer are aware of the distribution of the demand for the buyer's product.

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A Notation

b : Capacity of supplier

c_s : Cost of manufacturing the component

c_b : Cost of manufacturing product

D_t : Demand for buyer's product in period t

E : Starting inventory of component

h : Holding cost of component for supplier

h_b : Holding cost of product for buyer

L : Lower bound on order quantity for component

P_s : Price of component

P_b : Price of product

Q , Q_t : Order quantity for component in period t

s_b : Shortage cost of product to buyer

s : Shortage cost of component to supplier

U : Upper bound on order quantity for component i

X, X_t : Production decision in period t

μ_t : State variable representing the order quantity

$Q(\mu_t)$: Order quantity in period t

Π_b : Profit of the buyer

Π_s : Profit of the supplier

$\overline{\Pi}$: Minimum profit for the supplier

$K(.)$: One period profits for the supplier

I_{t-1} : Inventory at the beginning of period t for the supplier

Y_t : Inventory of supplier after prodution in period t

S_t : Critical number of supplier in period t

B_t : Inventory at beginning of period t for buyer

Z_t : Inventory of buyer after arrival of supply in period t

R_t : Critical number of buyer in period t

$J(.)$: One period profits for the buyer

B Proofs of Propositions

Proof of Proposition 1: This is a special case ($n = 1$) of Proposition 9. □

Proof of Proposition 2: This is a special case ($n = 1$) of Proposition 10. □

Proof of Proposition 3: We first note that any feasible production, X, must be such that

$$\begin{aligned} X + E &\leq U \\ X + E &\geq L \end{aligned} \tag{66}$$

This is evident because the supplier must at least produce to satisfy the minimum order, and no more than what would satisfy the maximum order. If the initial inventory is sufficient to cover the maximum order then the maxmin (or any solution) solution is not to produce. In the above range of values for X and for $h, s, c_s, P_s > 0$ we note the following:

- The profit at the minimum value of the order, L, is a monotonously decreasing function of X.
- The profit at the maximum value of the order, U, is a monotonously increasing function of X.

We define the range of values for X as (X_{min}, X_{max}) where

$$X_{max} = \min(\max(U - E, 0), b) \tag{67}$$

$$X_{min} = \max(L - E, 0) \tag{68}$$

We also note that $\Pi_s(X_{min}, 0) > \Pi_s(X_{min}, 1)$ and $\Pi_s(X_{max}, 0) < \Pi_s(X_{max}, 1)$. We may therefore conclude that X^* is such that $\Pi_s(X^*, 0) = \Pi_s(X^*, 1)$. Hence, we may compute the maxmin production as

$$P_s(X^* + E) - c_s X^* - (h + P_s)(X^* + E - L)^+ = P_s(X^* + E) - c_s X^* - s(X^* + E - U)^- \tag{69}$$

Hence

$$X^* = \min\left(\frac{sU + (h + P_s)L}{s + h + P_s} - E, 0\right) \tag{70}$$

Hence, the solution to the supplier's problem is

$$X = \begin{cases} X^* & \text{if } X_{min} \leq X^* \leq X_{max} \\ X_{min} & \text{if } X^* < X_{min} \\ X_{max} & \text{if } X^* > X_{max} \end{cases} \tag{71}$$

□

Proof of Proposition 4: We examine the first derivative as below

$$\frac{\partial \Pi_b}{\partial P_s} = (P_b - c_b - P_s + s_b)\frac{\partial X}{\partial P_s}[1 - F(X)] - E(Q) \tag{72}$$

where

$$E(Q) = [\int_0^L Lf(D)dD + \int_L^X Df(D)dD + \int_X^\infty Xf(D)dD] \tag{73}$$

$E(Q)$ represents the mean order quantity. Now

$$\frac{\partial X}{\partial P_s} = -\frac{2as}{s+h+P_s} \le 0 \tag{74}$$

Hence

$$\frac{\partial \Pi_b}{\partial P_s} \le 0 \tag{75}$$

□

Proof of Proposition 5: We define the variables $[Y_t]$ as the stock level after production (but before order placement) in the period. Hence

$$\begin{aligned} Y_t &= I_{t-1} + X_t \\ &= I_{t-1}^+ - I_{t-1}^- + X_t \\ I_t &= Y_t - Q(\mu_t) \end{aligned} \tag{76}$$

Note that $Y_t \ge I_{t-1}$ since $X_t \ge 0$. The recursive definitions are

$$\begin{aligned} f_t(I_{t-1}) &= \left[\max_{Y_t \ge I_{t-1}} \min_{\mu_t} K(Y_t, \mu_t) + f_{t+1}(Y_t - Q(\mu_t))\right], t = 1, \ldots, n \\ f_{n+1}(I_n) &= 0 \end{aligned} \tag{77}$$

where

$$K(Y_t, \mu_t) = P_s\,(Q(\mu_t) + I_{t-1}^-) - c_s(Y_t - I_{t-1}) - h(Y_t - Q(\mu_t))^+ - (s + P_s)(Y_t - Q\,(\mu_t))^- \tag{78}$$

Note that $Q(\mu_t) = L + \mu_t(U - L)$. Hence, it is clear that, for a given Y_t, $K(Y_t, \mu_t)$ is concave in $\mu_t, t = 1, \ldots, n$. Now, consider

$$f_n(I_{n-1}) = \max_{Y_n \ge I_{n-1}} \min_{\mu_n} K(Y_n, \mu_n) \tag{79}$$

$K(Y_n, \mu_n)$, from above, is concave in μ_n. Hence $\mu_n = 0$ or 1 minimizes $K(Y_n, \mu_n)$. Further, we observe that the function $K(Y_n, \mu_n = 0)$ monotonically and linearly decreases in $Y_n(Y_n \ge Q(0))$. Also the function $K(Y_n, \mu_n = 1)$ monotonically and linearly increases in $Y_n(Y_n \le Q(1))$. Hence, we conclude that $\min_{\mu_n} K(Y_n, \mu_n)$ is concave in Y_n and therefore, there exists a maximizing S_n. Now, let

$$G(Y_n) = P_s \min(Q(\mu_n), Y_n) - h(Y_n - Q(\mu_n))^+ - s(Y_n - Q(\mu_n))^- \tag{80}$$

That is, $G(Y_n)$ is the profit the supplier earns in period n before netting out the production cost. Since S_n maximizes $\min_{\mu_n} K(Y_n, \mu_n)$, we have

$$f_n(I_{n-1}) = \begin{cases} \min_{\mu_n} [G(S_n) - c_s(S_n - I_{n-1})] & \text{if } I_{n-1} \le S_n \\ \min_{\mu_n} [G(I_{n-1})] & \text{otherwise} \end{cases} \tag{81}$$

Note that the minimizing value of μ_n is 0 and 1 if $I_{n-1} \le S_n$ (since $K(S_n, \mu_n = 0) = K(S_n, \mu_n = 1)$), and 0 otherwise. Observe that $f_n(I_{n-1})$ is linear and increasing in I_{n-1} for $I_{n-1} \le S_n$. Further,

at $I_{n-1} = S_n, G(S_n) - c_s(S_n - I_{n-1}) = G(I_{n-1})$. Finally since $K(Y_n, \mu_n)$ is concave for a given $\mu_n, G(I_{n-1})$ is concave, and for $I_{n-1} > S_n$ decreasing in I_{n-1}. Hence $f_n(I_{n-1})$ is concave. We may now reason inductively. Let f_{t+1} be concave, and let S_{t+1} maximize f_{t+1}. Then we have

$$f_{t+1}(I_t) = \begin{cases} \min_{\mu_t} G(S_{t+1}) - c_s(S_{t+1} - I_t) + f_{t+2}(S_{t+1} - Q(\mu_{t+1})) & I_t \le S_{t+1} \\ \min_{\mu_t} G(I_t) + f_{t+2}(I_t - Q(\mu_t)) & \text{otherwise} \end{cases} \tag{82}$$

Hence $K(Y_t, \mu_t) + f_{t+1}(Y_t - Q(\mu_t))$ is concave, and therefore, is minimum at $\mu_t = 0$ or 1. Now, at $\mu_t = 0$, for $Q(0) \le Y_t < Q(0) + S_{t+1}$

$$\begin{aligned} & K(Y_t, \mu_t) + f_{t+1}(Y_t - Q(\mu_t)) \\ &= K(Y_t, \mu_t) + G(S_{t+1}) - c_s(S_{t+1} - Y_t + Q(\mu_t)) + f_{t+2}(S_{t+1} - Q(\mu_{t+1})) \\ &= G(Y_t) + G(S_{t+1}) - c_s(S_{t+1} - I_{t-1} + Q(0)) + f_{t+2}(S_{t+1} - Q(\mu_{t+1})) \end{aligned} \tag{83}$$

Note that the last three terms are constant, while $G(Y_t)$ is decreasing linearly for increasing Y_t at a rate h. Further, for $Y_t - Q(0) > S_{t+1}$, we have

$$K(Y_t, \mu_t) + f_{t+1}(Y_t - Q(\mu_t)) = K(Y_t, \mu_t) + G(Y_t - Q(\mu_t)) + f_{t+2}(Y_t - Q(\mu_t) - Q(\mu_{t+1})) \tag{84}$$

Observe that $G(Y_t - Q(\mu_t)) + f_{t+2}(Y_t - Q(\mu_t) - Q(\mu_{t+1}))$ is concave decreasing in Y_t. Further, $K(Y_t, \mu_t)$ decreases at the rate $c_s + h$ for increasing Y_t. Hence $K(Y_t, \mu_t) + f_{t+1}(Y_t - Q(\mu_t))$ is concave decreasing in Y_t. For $\mu_t = 1$, and $Y_t \le Q(1)$

$$K(Y_t, \mu_t) + f_{t+1}(Y_t - Q(\mu_t)) = G(Y_t) + G(S_{t+1}) - c_s(S_{t+1} - I_{t-1} + Q(1)) + f_{t+2}(S_{t+1} - Q(\mu_{t+1})) \tag{85}$$

$G(Y_t)$ increases at the rate $s + P_s$. Hence $K(Y_t, \mu_t) + f_{t+1}(Y_t - Q(\mu_t))$ increases linearly.

Hence $\min_{\mu_t} K(Y_t, \mu_t) + f_{t+1}(Y_t - Q(\mu_t))$ is concave and there exists a maximizing S_t. We have

$$f_t(I_{t-1}) = \begin{cases} \min_{\mu_t} [G(S_t) - c_s(S_t - I_{t-1}) + f_{t+1}(S_t - Q(\mu_t))] & \text{if } I_{t-1} \le S_t \\ \min_{\mu_t} [G(I_{t-1}) + f_{t+1}(I_{t-1} - Q(\mu_t))] & \text{otherwise} \end{cases} \tag{86}$$

As earlier, we note that the minimizing value of $\mu_t = 0$ or 1 for $I_{t-1} \le S_t$, and 0 otherwise. This, as before, leads us to the conclusion that $f_t(I_{t-1})$ is linear increasing for $I_{t-1} \le S_t$ and that it is concave decreasing for I_{t-1} Hence f_t is concave. The proof is complete. □

Proof of Proposition 6: Let

$$\begin{aligned} W_t &= Z_t - D_t \\ H(W_t) &= -h_b(W_t)^+ - (P_s + s_b)(W_t)^- \\ \text{Hence } J(W_t) &= (P_b - c_b)D_t - P_s(Z_t - B_t^+) + H(W_t) \end{aligned} \tag{87}$$

We note

1. $H(W_t)$ is concave in W_t.
2. $\lim_{|W_t| \to \infty} H(W_t) \to -\infty$.

Consider period n

$$f_n(B_n) = \max(P_b - c_b)D_n - P_s(Z_n - B_n^+) + H(W_n) \tag{88}$$

From 1 and 2 above there exists a maximizing value of Z_n, and let this value be $D_n + R_n$. Therefore

$$f_n(B_n) = \begin{cases} (P_b - c_b)D_n - P_s(S_n) + H(S_n + B_n^+ - D_n) & S_n + B_n^+ < D_n + R_n \\ (P_b - c_b)D_n - P_s(D_n + R_n - B_n^+) + H(R_n) & L < D_n + R_n - B_n^+ \leq S_n \\ (P_b - c_b)D_n - P_s L + H(L + B_n^+ - D_n) & D_n + R_n - B_n^+ \leq L \end{cases} \tag{89}$$

Therefore $f_n(B_n)$ is concave. Now we reason inductively. We have

$$f_t(B_t) = \max(P_b - c_b)D_t - P_s(Z_t - B_t^+) + H(W_t) + E(f_{t+1}(W_t)) \tag{90}$$

Let f_{t+1} be concave. Then clearly the terms in the argument of the max operand above are concave. Hence there exists a maximizing number R_t. Given the critical number R_t, we have

$$f_t(B_t) = \begin{cases} (P_b - c_b)D_t - P_s S_t + H(S_t + B_t^+ - D_t) & \\ +E(f_{t+1}(S_t + B_t^+ - D_t)) & S_t + B_t^+ < D_t + R_t \\ (P_b - c_b)D_t - P_s(D_t + R_t - B_t^+) + H(R_t) + E(f_{t+1}(R_t)) & L < D_t + R_t - B_t^+ < S_t \\ (P_b - c_b)D_t - P_s L + H(L + B_t^+ - D_t) & \\ +E(f_{t+1}(L + B_t^+ - D_t)) & D_t + R_t - B_t^+ < L \end{cases} \tag{91}$$

$f_t(B_t)$ is therefore concave. Hence, the proof is complete. □

Proof of Proposition 7: The recursive definitions are

$$\begin{aligned} f_t(I_{t-1}) &= max_{Y_t} min_{\mu_t} K(Y_t, \mu_t) + f_{t+1}(Y_t - Q_(\mu_t)) \\ f_{n+1}(I_n) &= 0 \end{aligned} \tag{92}$$

where

$$K(Y_t, \mu_t) = P_s(Q(\mu_t) + I_{t-1}^-) - c_s(Y_t - I_{t-1}) - h(Y_t - Q(\mu_t))^+ - (s + P_s)(Y_t - Q(\mu_t))^- \tag{93}$$

It has been shown in Proposition 5 that $K(Y_t, \mu_t)$ is concave in μ_t. Consider

$$f_n(I_{n-1}) = \max_{Y_n} \min_{\mu_n} K(Y_n, \mu_n) \tag{94}$$

Since $K(Y_n, \mu_n)$ is concave in μ_n, $\mu_n = 0$ or 1 minimizes $K(Y_n, \mu_n)$. As in Proposition 5, it may be shown that $\min_{\mu_n}(K(Y_n, \mu_n))$ is concave in Y_n, and there exists a maximimizing S_n. As before, define

$$G(Y_n) = P_s \min(Q(\mu_n), Y_n) - h(Y_n - Q(\mu_n))^+ - s(Y_n - Q(\mu_n))^- \tag{95}$$

Since S_n maximizes $min_{\mu_n} K(Y_n, \mu_n)$.

$$f_n(I_{n-1}) = \begin{cases} \min_{\mu_n} [G(I_{n-1} + b) - c_s b] & \text{if } I_{n-1} + b < S_n \\ \min_{\mu_n} [G(S_n) - c_s(S_n - I_{n-1})] & \text{if } I_{n-1} \leq S_n \leq I_{n-1} + b \\ \min_{\mu_n} [G(I_{n-1})] & \text{otherwise} \end{cases} \tag{96}$$

The minimizing value of μ_n is 0 if $I_{n-1} > S_n$, is 0 and 1 if $I_{n-1} \leq S_n \leq I_n + b$, and is 1 if $I_{n-1} + b \leq S_n$. Observe that $f_n(I_{n-1})$ is linear and increasing in I_{n-1} at a rate $P_s + s$ for $I_{n-1} + b < S_n$ and a rate c_s for $I_{n-1} \leq S_n \leq I_n + b$. Further, at $I_{n-1} + b = S_n$, $G(S_n) - c_s(S_n - I_{n-1}) = G(I_{n-1} + b) - c_s b$. Further, as in Proposition 5, $G(I_{n-1})$ is decreasing concave in I_{n-1} for $I_{n-1} \geq S_n$. Hence, $f_n(I_{n-1})$ is concave. Further, note that f_n is increasing at a rate $P_s + s > c_s$ in the region $I_{n-1} + b > S_n$.

We now reason inductively. Let f_{t+1} be concave and let S_{t+1} maximize f_{t+1}. Then

$$f_{t+1}(I_t) = \begin{cases} \min_{\mu_t} [G(I_t + b) - c_s b + f_{t+2}(I_t + b - Q(\mu_{t+1}))] & \text{if } I_t + b < S_{t+1} \\ \min_{\mu_t} [G(S_{t+1}) - c_s(S_{t+1} - I_t) + f_{t+2}(S_{t+1} - Q(\mu_{t+1}))] & \text{if } I_t \le S_{t+1} \le I_t + b \\ \min_{\mu_t} [G(I_t) + f_{t+2}(I_t - Q(\mu_{t+1}))] & \text{otherwise} \end{cases} \quad (97)$$

Let f_{t+1} be increasing in I_t at a rate greater than c_s when $I_{t+1} + b < S_{t+1}$. Hence $K(Y_t, \mu_t) + f_{t+1}(Y_t - Q(\mu_t))$ is concave and hence minimum at $\mu_t = 0$ or 1. At $\mu_t = 0$, two cases are considered

Case 1: $S_{t+1} > b$. For $Q(0) \le Y_t \le S_{t+1} - b + Q(0)$
Note that $K(Y_t, \mu_t)$ is decreasing in Y_t at the rate $h + c_s$. Further, in this region, f_{t+1} is concave increasing at a rate greater than c_s. Hence, f_{t+1} is either concave increasing or concave decreasing at a rate less than h. For the regions $S_{t+1} - b + Q(0) \le Y_t \le S_{t+1}$ and $Y_t > S_{t+1}$ the analysis is similar to the analysis in Proposition 5. Hence, overall, $K(Y_t, \mu_t) + f_{t+1}(Y_t - Q(\mu_t))$ is concave in Y_t.

Case 2 $b \ge S_{t+1}$ There are only two regions here and these correspond to the last two regions considered in Case 1. These are similar to the analysis in Proposition 5. In this case, $K(Y_t, \mu_t) + f_{t+1}(Y_t - Q(\mu_t))$ is concave decreasing in Y_t.

For $\mu_t = 1$, consider the region $Y_t \le S_{t+1} + Q(1) - b$

$K(Y_t, \mu_t)$ is increasing at the rate $s + P_s$ and f_{t+1} is also concave increasing. Hence, $K(Y_t, \mu_t) + f_{t+1}(Y_t - Q(\mu_t))$ is concave increasing at a rate higher than $s + P_s$. For the other possible region $Y_t > S_{t+1} + Q(1) - b$, it is shown in Proposition 5 that the function is increasing at the rate $s + P_s$. Hence, $K(Y_t, \mu_t) + f_{t+1}(Y_t - Q(\mu_t))$ is concave increasing. Hence $\min_{\mu_t} K(Y_t, \mu_t) + f_{t+1}(Y_t - Q(\mu_t))$ is concave and there exists a maximizing S_t. Hence

$$f_t(I_{t-1}) = \begin{cases} \min_{\mu_{t-1}} [G(I_{t-1} + b) - c_s b + f_{t+1}(I_{t-1} + b - Q(\mu_t))] & \text{if } I_{t-1} + b < S_t \\ \min_{\mu_{t-1}} [G(S_t) - c_s(S_t - I_{t-1}) + f_{t+1}(S_t - Q(\mu_t))] & \text{if } I_{t-1} \le S_t \le I_{t-1} + b \\ \min_{\mu_{t-1}} [G(I_{t-1}) + f_{t+1}(I_{t-1} - Q(\mu_t))] & \text{otherwise} \end{cases} \quad (98)$$

The minimizing value of μ_t is 1 in the first region, 0 and 1 in the second region, and 0 in the third region. Hence, it is easily verified that f_t is concave. It is also easily verified that, in the first region, f_t is increasing at rate greater than c_s. The proof is complete. □

Proof of Proposition 8: Since $G(X')$ is optimal we have

$$G(X') \ge \min_{\mu} G(X^*, \mu) = G(X^*, \mu' = F(X^*, \mu') \ge F(X^*) \quad (99)$$

□

Proof of Proposition 9:

$$\begin{aligned} F(X, \mu) &= \sum_t K(I_{t-1}, X_t, \mu_t) \\ &= \sum_t P_s \min\left(X_t + I_{t-1}^+, Q(\mu_t) + I_{t-1}^-\right) - c_s - hI_t^+ - sI_t^- \\ &= \left[\sum_t \left[P_s Q(\mu_t) - c_s X_t - hI_t^+ - sI_t^-\right] - PsI_n^-\right] \end{aligned} \quad (100)$$

Note $\sum_t P_s Q(\mu_t) - c_s X_t$ is concave in μ. Further,

$$\begin{aligned} P_s I_n^- &= P_s \left[I_{n-1}^- + Q(\mu_n) - I_{n-1}^+ - X_n\right]^+ \\ &= P_s \max\left[0, I_{n-1}^- + Q(\mu_n) - I_{n-1}^+ - X_n\right] \end{aligned} \tag{101}$$

is convex in μ.

We now need to show that $\sum_t (hI_t^+ + sI_t^-)$ is convex. We note first that $Q(\mu_t)$ is a linear function of μ (Equation 1). Now

$$\begin{aligned} I_t^+ - I_t^- &= X_t + I_{t-1}^+ - Q(\mu_t) - I_{t-1}^- \\ &= \sum_{t'=1}^{t} (X_{t'} - Q(\mu_{t'})) + I_0 \end{aligned} \tag{102}$$

Hence $I_t^+ - I_t^-$ is a linear function of μ.

Now

$$\begin{aligned} I_t^+ &= \max(0, X_t - Q(\mu_t^+)\, I_{t-1}^+ - I_{t-1}^-) \\ &= \max\left(0, \sum_{t'=1}^{t-1} (X_{t'} - Q(\mu_{t'})) + I_0\right) \end{aligned} \tag{103}$$

Hence I_t^+ is the maximum of two convex functions and hence convex.

Similarly

$$\begin{aligned} I_t^- &= \max(0, -(X_t - Q(\mu_t^+)\, I_{t-1}^+ - I_{t-1}^-)) \\ &= \max\left(0, -\left(\sum_{t'=1}^{t-1} (X_{t'} - Q(\mu_{t'})) + I_0\right)\right) \end{aligned} \tag{104}$$

Hence I_t^- is the maximum of two convex functions and hence convex

Hence, $\sum_t (hI_t^+ + sI_t^-)$ is convex.

This concludes the proof. □

Proof of Proposition 10: This follows immediately from the concavity of $F(X,\mu)$. See Luenberger (1973).

Proof of Proposition 11: Choose j such that

$$F(X,\mu^j) \le F(X,\mu^k) \text{ for } 1 \le j,k \le |\Phi|, j \ne k \tag{105}$$

Clearly $F(X^*,\mu^j) = z$, where z is the optimal function value to the linear program.

Suppose $E_t^j, S_t^j > 0$. Then there exists an optimal solution to the linear program such that $E_t^j > 0$ and $S_t^j > 0$. Let $\epsilon \le \min(E_t^j, S_t^j)$. We shall subtract ϵ from E_t^j, S_t^j and investigate the changes that must be made to the other variables such that the constraints in the LP hold.

The constraints are

$$E_{t+1}^j - S_{t+1}^j = X_{t+1} + E_t^j - Q\left(\mu_{t+1}^j\right) - S_t^j$$
$$E_t^j - S_t^j = X_t + E_{t-1}^j - Q\left(\mu_t^j\right) - S_{t-1}^j \tag{106}$$

Note that the two equations above are unaffected by the proposed changes. The total change to the profit $F(X^*, \mu^j)$ is given by

$$(h+s)\epsilon \tag{107}$$

Since, in the optimal solution, this constraint is binding, we infer that the objective function value may be increased by the same amount. While doing this increase on the profit at the extreme point j, it is possible that the profit at j may become larger than at some other extreme point k. In this case, we perform the above operations on some pair of variables in the profit function $F(X^*, \mu^k)$. If no such pair of variables $E_t^k > 0, S_t^k > 0$ exist, then the proof is complete. □

Proof of Proposition 12: Note that the domain of feasible decisions X is the same in both the linear and the nonlinear formulations, since the same capacity constraint exists in both formulations.

If X^*, z^* are the optimal solutions to the linear program, the by Proposition 11 we have

$$\sum t\left[P_s Q\left(\mu_t^j\right) - c_s X_t^* - hE_t^j sS_t^j\right] - P_s S_n^j = F(X^*, \mu^j) = z^*$$
$$\sum t\left[P_s Q\left(\mu_t^k\right) - c_s X_t^* - hE_t^k sS_t^k\right] - P_s S_n^k \geq F(X^*, \mu^k) \geq z^* \tag{108}$$

for some j and for every other $k, k \neq j$. Hence X^*, z^* is feasible in the nonlinear program. It must also be optimal, since every other feasible solution to the nonlinear program is also feasible in the linear program.

Since every solution to the nonlinear program is feasible to the nonlinear program, and from above the optimal solution to the linear program is feasible in the nonlinear program, we conclude that the optimal solution to the nonlinear program must be feasible and optimal in the linear program. □

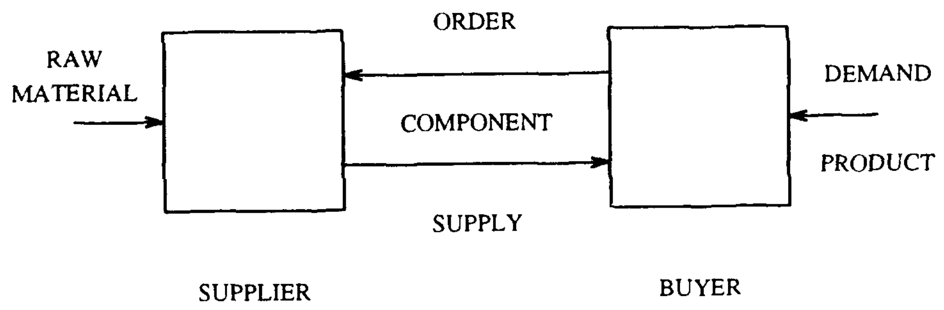

FIGURE 1: STRUCTURE OF THE BUYER SUPPLIER RELATIONSHIP

FIGURE 2 : THE MINIMUM COMPONENT PRICE FOR EACH ORDER RANGE

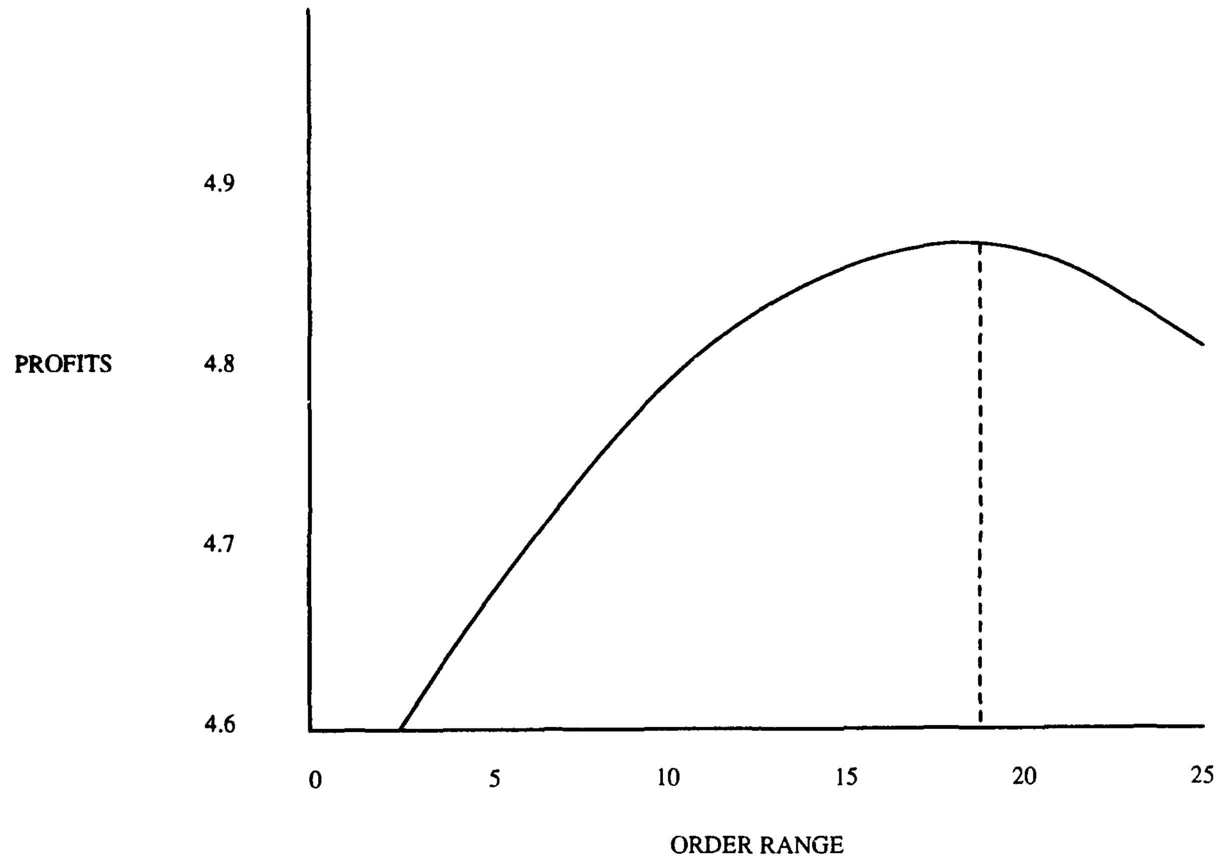

FIGURE 3 : THE OPTIMAL PROFITS OF THE BUYER

FIGURE 4 : MINIMUM SUPPLIER PRICE

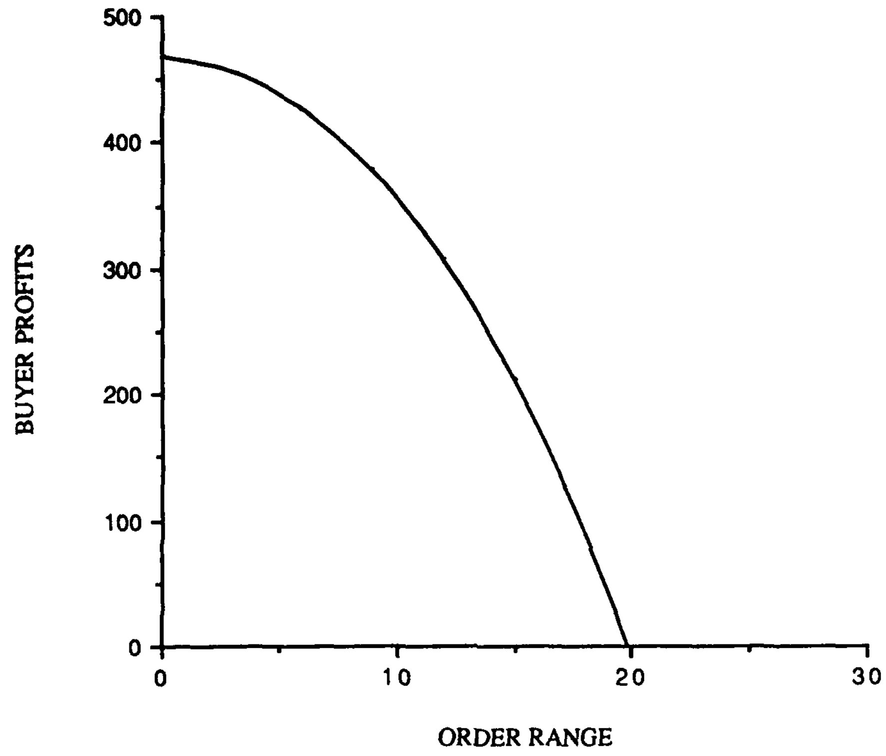

FIGURE 5 : THE OPTIMAL PROFITS OF THE BUYER

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It is the practice in some industries, as well as within multiplant organizations, to specify bounds on the range of allowable order values. That is, the buyer contracts to place an order within a small range in a future period. The specification of the range protects the supplier against large variations in the order at short notice, although it reduces the flexibility of the buyer to respond to demand changes in his own market. Hence in exchange, the supplier is willing to reduce the price of the component to the buyer. This paper examines the implications of this specification for the production/inventory decisions. The optimal contract is determined from the structure of the solution to the